SUFFICIENT CONDITIONS FOR THE EXISTENCE OF UNIQUE SOLUTIONS OF INVERSE BOUNDARY VALUE PROBLEMS

(DOSTATOCHNYE USLOVIIA ODNOLISTNOSTI RESHENIIA OBRATNYKH KRAEVYKH ZADACH)

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V. S. ROGOZHIN
(Rostov-on-Don)

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1. In solving the inverse problem of hydromechanics, which consists in the determination of an airfoil section (profile) on the basis of the distribution of the velocities of the flow of an ideal fluid around the airfoil, there arises the question of the uniqueness (single-sheetedness) of the profile. Since multiple-sheeted solutions of this problem are meaningless, it follows that the conditions of uniqueness are conditions for the possibility of solving the problem.

It is well known that uniqueness of the solution of the problem under consideration is equivalent to the uniqueness of the function

$$z = z(\zeta) = a_0\zeta + \sum_{k=0}^{\infty} \frac{a_k}{\zeta^k}$$

whose derivative has the form

$$\frac{dz}{d\zeta} = \exp\left[-\frac{1}{2\pi}\int_0^{2\pi} p(\theta)\,\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\,d\theta\right] \qquad (|\zeta| > 1) \tag{1}$$

where $p(\theta)$ is expressed in terms of the given distribution of velocities along the profile.

We denote the lower bound of the lengths of smooth arcs lying in a given region and joining two points ζ_1 and ζ_2, by $l(\zeta_1, \zeta_2)$.

Lemma. If $f(\zeta)$ is regular in the infinite region G for which

$$\sup \frac{l(\zeta_1, \zeta_2)}{|\zeta_1 - \zeta_2|} = \lambda_0 \neq \infty \text{ in the region } G,$$

except at a simple pole, at $\zeta = \infty$, and if the set of values of $f'(\zeta)$ lies within the circle with center at $\zeta = a$, and of radius $|a| / \lambda_0$, then the function $f(\zeta)$ is unique.

Proof. We have

$$|f(\zeta_1) - f(\zeta_2)| > |\zeta_1 - \zeta_2| \left[|a| - \max |f'(\zeta) - a| \left| \frac{l(\zeta_1, \zeta_2)}{|\zeta_1 - \zeta_2|} \right| \right]$$

and since

$$\sup \frac{l(\zeta_1, \zeta_2)}{|\zeta_1 - \zeta_2|} = \lambda_0, \qquad |f'(\zeta) - a| < \frac{|a|}{\lambda_0}$$

it follows that the expression in square brackets is positive, and that

$$|f(\zeta_1) - f(\zeta_2)| > 0$$

Remark: For the exterior of the circle, $\lambda_0 = \pi/2$.

Theorem 1. If

1) $$p(\theta) = p_1(\theta) + \ln |e^{i\theta} - 1|^{\alpha}$$

2) $$|p_1(\theta_1) - p_1(\theta_2)| < k |\theta_1 - \theta_2|$$

3) $$\max |e^{i\theta} - 1|^{\alpha} e^{p_1(\theta)} < \frac{(\pi \cos \beta + \sqrt{4 - \pi^2 \sin^2 \beta})^2}{\pi^2 - 4}$$

$$\left(\beta = 2k \ln 2 + \tfrac{1}{2} \alpha \pi\right)$$

then the function $z(\zeta)$, whose derivative is of the form (1), will be unique outside the unit circle.

Proof. From hypotheses 1 and 2 it follows that

$$|\arg f'(\zeta)| \leqslant \alpha \frac{\pi - \theta}{2} + \left| \frac{1}{2\pi} \int_0^{2\pi} p_1(\theta^\circ) \cot \frac{\theta - \theta^\circ}{2} \, d\theta^\circ \right| < \alpha \frac{\pi}{2} + 2k \ln 2 = \beta$$

This implies that the values $w = f'(\zeta)$ lie inside the curvilinear trapezoid

$$|\arg f'(\zeta)| < 2k \ln 2, \qquad \exp P_{\min} < |f'(\zeta)| < \exp P_{\max}$$

which on the basis of hypothesis 3 can be shown to lie within some circle with center at $a > 0$, and of radius $2a/\pi$. The truth of the theorem now follows from the Lemma and the above remark.

Making use of this theorem, we can derive the conditions the given velocity (as a function of arc length s) must satisfy along the desired profile for the profile (airfoil cross-section) to prove unique. For this purpose we introduce an auxiliary fluid flow which flows around a unit circle L_0 with the same velocity at infinity and the same circulation as those of the flow around the solution profile L. We put the points of L_0

and of L which have equal velocity potentials into a one-to-one correspondence. The ratio v_0/v, where v_0 is the velocity on the circle L_0, while v is the velocity on the profile, can now be expressed as a function of the central angle on the circle L_0.

It is known that the z-plane region, occupied by the flow around the required profile, can be obtained by mapping the exterior of the unit circle in the ζ-plane by means of an analytic function $Z(\zeta)$ whose derivative has the form (1) with $p(\theta) = n(v_0/v)$. Theorem 1 now gives us the sufficient conditions for the uniqueness of the solution of the inverse problem. It is clear that for the application of Theorem 1 with $a = 0$, it is necessary that v_0/v be a function of θ which is finite and different from zero. One can show that these conditions will be fulfilled if the velocity $v(s)$, given on the profile, has the form: $v(s) = s(s - s_B)\ v_1(s)$, where $v_1(s)$ is different from zero when $s = 0$ and when $s = s_B$ ($s = 0$ is a stagnation point of the flow, and s_B is the convergence point, rear edge point).

It is known [1] that in this case the resulting profile will be smooth. On other assumptions with respect to the type of the function $v(s)$, the resulting profile can have corner points, and the ratio v_0/v can take on the value zero.

Suppose, for example, that we are looking for a profile (airfoil section) which is smooth at the stagnation point, while at the rear edge point its tangents form an angle $\delta\pi$. In this case the velocity v can be represented in the form

$$v = |e^{i\theta} - 1|^{\delta}\, |e^{i\theta} + 1|\, f_0(\theta)$$

where $f_0(\theta)$ is finite and different from zero.

The function $p(\theta) = \ln(v_0/v)$ takes the form

$$\ln v_0 \,|\, v = \ln |{}^{i\theta} - 1\, e^{1-\delta} + p_1(\theta)$$

where $p_1(\theta)$ is continuous and on certain additional assumptions will satisfy the Lipschitz condition. Theorem 1 will give a sufficient condition for the uniqueness of the resulting profile in this case if $a = 1 - \delta$.

The derived criteria for uniqueness are especially convenient to use, if the velocity along the required profile is given not as a function of the arc length s of the profile, but as a function of the points on the auxiliary circle L_0, as was proposed by Peebles [2].

Let us next consider the problem of improving the properties of the given profile while preserving its uniqueness. Thus, let the profile S_0

be given. It is assumed to be smooth, and at the point of convergence (rear edge point) B, it has two tangents making an angle $a\pi$ $(a \neq 0)$. The distribution of the velocity $v = f_0(s)$ along S_0 is replaced by a new function $v = f(s)$; it is required to determine the conditions that will guarantee the uniqueness of the new profile S. The length of the profile will be assumed to remain constant, while the functions $f_0(s)$ and $f(s)$ coincide at the stagnation point $s = 0$ and the convergence point $s = s_B$. Let us assume that the given profile S_0 lies in the Z-plane and the resulting profile S in the z-plane. Let us place the circle $|\zeta| = 1$ in the ζ-plane.

Then

$$\frac{dz}{dZ} = \exp \frac{1}{2\pi} \int_0^{2\pi} p(\theta) \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\theta \tag{1}$$

where $p(\theta) = \ln |dz/dZ| = \ln(v_0/v)$ is considered as a function of the polar (central) angle of the unit circle. Obviously, the resulting profile will be unique if the function $z = z(Z)$ is unique.

Let us suppose that in the exterior of S_0 the following inequality is valid:

$$\sup \frac{l(z_1, z_2)}{|z_1 - z_2|} = \lambda_0 \neq \infty \quad \text{for } z_1, z_2 \text{ in the region } G.$$

Then the following theorem holds.

Theorem 2. If

$$1)\ |p(\theta_1) - p(\theta_2)| < k|\theta_1 - \theta_2|$$

$$2)\ \max \frac{v_0}{v} \Big/ \min \frac{v_0}{v} < \frac{(\lambda_0 \cos\beta + \sqrt{1 - \lambda_0^2 \sin^2\beta})^2}{\lambda_0^2 - 1}$$

$$(\beta = 2k \ln 2)$$

then the function $z(Z)$ is unique outside S_0.

The proof of this theorem is similar to that of Theorem 1.

2. The uniqueness of the solution of the inverse problem of the theory of filtration [1] is equivalent to the uniqueness of the function $z = z(t)$, which is analytic in the lower half-plane and whose derivative has the form

$$\frac{dz}{dt} = \exp\left[-\frac{\sqrt{1-t^2}}{\pi i} \int_{-1}^{1} p(\tau) \frac{d\tau}{\sqrt{1-\tau^2}(\tau - t)}\right] \tag{2}$$

where the function $p(\tau)$ is determined by the data given in the statement of the problem.

Theorem 3. If

$$p(\tau) = -p_1(\tau) - \ln(1-\tau)^p - \ln(1+\tau)^q$$
$$|p_1(\tau_1) - p_1(\tau_2)| < \frac{1}{2}\pi(1-p-q)$$

then the function $z(t)$, whose derivative has the form (2), is unique.

Proof. Using the formula

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\ln(1-\tau)^p(1+\tau)^q}{\sqrt{1-\tau^2}(\tau-\xi)}\,d\tau = \frac{p+q}{\sqrt{1-\xi^2}}\cos^{-1}\xi - \frac{p\pi}{\sqrt{1-\xi^2}}$$

we can show that the oscillation of the contour values of the argument of the function dz/dt is less than π. This, however, is the well-known [3] condition for the uniqueness of the function $z(t)$.

If the given velocity $v(s)$ of the inverse problem has the form

$$v(s) = s^{n_1}(l-s)^{n_2} f_0(s) \tag{3}$$

where $f_0(s)$ is different from zero in $0 \leqslant s \leqslant l$, then the angles formed by the resulting contour with the horizontal at the point $s = 0$ and $s = l$ will be

$$\alpha_1 = \frac{1}{2(1+n_1)}, \qquad \alpha_2 = \frac{1}{2(1+n_2)} \tag{4}$$

respectively, and the function $v(s)$ can be represented in the form

$$v(s) = f_0(s)\left[1 + \sin\frac{\pi\varphi(s)}{kH}\right]^{1/2-\alpha_1}\left[1 - \sin\frac{\pi\varphi(s)}{kH}\right]^{1/2-\alpha_2}$$

Here $f_0(s) \neq 0$, and the potential $\phi(s)$ is found by the following formula:

$$\varphi(s) = \int_0^s v(s)\,ds - \frac{kH}{2}$$

(H is the pressure, k is the coefficient of filtration)

The function which maps the region (of the complex potential plane w) corresponding to the resulting flow upon the lower half-plane, Im $\zeta < 0$, has the form

$$w = \frac{kH}{\pi}\sin^{-1} t \tag{5}$$

Hence,

$$p(t) = \ln\left|\frac{dw}{d\tau}\;\frac{dz}{dw}\right| = -\ln f_0 - (1-\alpha_1)\ln(1+t) - (1-\alpha_2)\ln(1-t)$$

The condition for uniqueness of the solution of the inverse problem of the theory of filtration can be obtained by putting

$$p = 1 - \alpha_1, \qquad q = 1 - \alpha_2, \qquad p_1(t) = -\ln f_0[s(t)]$$

in the hypotheses of Theorem 3.

This condition can also be written in a somewhat different form if we take into account the fact that $t = \sin[\pi\phi(s)/kH]$.

It is interesting to note the physical meaning of the criterion obtained for uniqueness. Let us take into consideration an auxiliary flow under a planar spillway $-1 < t < 1$ with the same pressure as in the problem under consideration; then $p(t)$ can be considered as the logarithm of the ratio of the velocity on the auxiliary planar spillway to the velocity on the resultant at corresponding points (here two points on the two spillways are said to correspond to each other if the velocity potentials are equal at these points).

With this arrangement, the condition of single-sheetedness in Theorem 3 can be interpreted as the requirement that the given velocity and the velocity on the planar spillway be comparable, in a certain sense, with each other.

Let us now return to the problem of improving the properties of the known contour.

We first introduce some definitions. Let B be a region bounded by a stepwise smooth curve. Let $\delta_{z1, z2}$ be the lower bound of the oscillation of the argument dz over all possible smooth curves lying on B and joining z_1 and z_2, and let $\delta_B = \sup \delta_{z1, z2}$ where the upper bound is taken over all pairs of points (z_1, z_2) belonging to B.

Theorem 4. If the function $f(z)$, which is regular in the region B, for which $\delta_B < \pi$, has a derivative in the closed region B, and if the oscillation of the argument of this derivative, i.e. $\Delta \arg f'(t)$, on the boundary of the region satisfies the inequality

$$\Delta_L \arg f'(t) \leqslant \rho < \pi - \delta$$

then $f(z)$ is unique in B.

Proof. Having selected the path of integration appropriately, we can prove that for any pair of points z_1 and z_2

$$\operatorname{Im}\int_{z_1}^{z_2} f'(z)\,dz = \operatorname{Im}[f(z_1) - f(z_2)] \neq 0 \tag{6}$$

as was required.

Let L_0 be a given contour, lying in the plane Z, whose properties are to be improved. Let L be the improved contour lying in the plane z. We denote the region occupied by the flow in the Z-plane by B_0, and designate the flow region in the z-plane by B. Moreover, we assume that $\delta_{B_0} < \pi$.

On the basis of Theorem 4 we can make the following assertion. For the uniqueness of the resultant region B it is sufficient for the oscillation of the argument of the function dz/dZ to be not greater than $\pi - \delta$ in B_0.

On L_0 we have

$$|dz/dZ| = v/v_0$$

where v is the velocity on the resultant contour while v_0 is the velocity on the given contour.

Let us consider dz/dZ as a function of the variable t which varies over the lower half-plane Im $t < 0$. Making use of the Keldysh-Sedov formula we obtain

$$\frac{dz}{dZ} = \exp \chi(t) \qquad \left(\chi(t) = -\frac{\sqrt{1-t^2}}{\pi i} \int_{-1}^{1} \ln \frac{v}{v_0} \frac{d\tau}{\sqrt{1-\tau^2}(\tau - t)}\right)$$

Let us assume that the change of velocity takes place only at the interior points of L_0, while v and v_0 are different from zero for all s such that $0 < s < l$; then the function $\ln(v/v_0)$ will be continuous if v and v_0 are continuous.

Next, let the following inequality be satisfied for the function $p(t) = \ln(v/v_0)$

$$|p(t_1) - p(t_2)| < \frac{1}{2}(\pi - \delta)|t_1 - t_2|$$

We then obtain the following bound on the oscillation of the function $\arg dz/dt = Q(t)$

$$\Delta_z Q(t) < \pi - \delta$$

From the theorem on the maximum of an analytic function it follows that this bound also holds in the interior of the region, and thus guarantees the uniqueness of the resultant region.

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